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# Noncommuting spinor rotation due to balanced geometrical and dynamical phases

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A neutron polarimetric textbook experiment is performed to demonstrate the noncommutation properties of the Pauli spin operator. The polarization of the transmitted neutron beam depends upon the sequential order of two successive spin-flip devices with mutually inclined precession axes. Furthermore, an interpretation of the observed results is presented in terms of an intrinsic phase shift between two interfering mutually orthogonal spin states, which in a sequence of two successive spinor transformations stores particular information about the intermediate state. This phase shift is discussed with respect to its geometrical and dynamical components, and its formal connection with the noncommuting properties of Pauli spin operators is explicitly shown. [S1050-2947(99)01806-5]

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## I. INTRODUCTION

Mathematically the rotation of spin-$\frac{1}{2}$ particles by a magnetic field is described by the action of one of the simplest yet most important operators in quantum mechanics, namely the Pauli spin operator, $\boldsymbol{\sigma}$ [1]. It is well known that the orthogonal components of this Pauli spin operator are mutually complementary observables, i.e., they do not commute with each other. This concept of noncommuting operators is one of the most fundamental principles of quantum mechanics [1–3] and it is closely related to Heisenberg's famous uncertainty principle, which has been interpreted either as indeterminacy [4] or as complementarity [5].

Neutron optical experiments, in particular those where interference effects of matter waves are involved, have served as elegant demonstrations of the foundations of quantum mechanics [6–9]. Among its other intrinsic properties, it is its half-integer spin that makes the neutron an ideal object for quantum physical textbook experiments [10]. The first explicit experimental verification of the $4\pi$ periodicity of spinor wave functions [11–13], the demonstration of static and time-dependent superposition of fermion spin states [14,15], and magnetic resonance-induced macroscopic quantum beating [16] were the highlights of a series of neutron interference experiments which have given new insights into the foundations of quantum physics. Recently neutron polarimetry has turned out to serve as another tool to unambiguously verify some of the basic concepts of quantum mechanics, such as, for instance, the topological nature of the scalar Aharonov-Bohm effect [17], where any general incident polarization is interpreted as the superposition of two orthogonal spin states. Different phase shifts of these two states due to interaction with a magnetic field lead to a polarization change of the outgoing neutron beam [18]. This implicit neutron polarization interference scheme will allow us to perform a textbooklike demonstration of the commutation relations of the Pauli spin operator and to clarify the connection between neutron spin rotations and the associated geometric and dynamical phase shifts of the wave function.

Since its first clear description by Berry [19], the geometric effect on the phase of the wave function has found considerable interest [20]. The geometric phase is closely related to the phase discovered by Pancharatnam [21] in the 1950s and it became manifest in a series of experiments [22], among which were a few involving neutron spin rotations during adiabatic passage through varying magnetic fields [23,24]. In addition, the geometric effect associated with cyclic evolutions in Hilbert space obeying SU(2) algebra was revealed [25]. A fundamental relation was also shown between quantum-mechanical backreaction in measurements and the Berry effect [26]. Quite recently it was argued that the noncommuting properties of two identical successive neutron spin flippers with different orientations of their precession axes will lead to an observable geometric phase shift effect [27], which has indeed been verified both by neutron interferometry [28,29] and polarimetry [30].

In this paper noncommuting spinor rotations are demonstrated by means of a neutron polarimetric setup. The incident polarized neutron beam passes sequentially through two $\pi$ spin-turn devices which are identical except for a different orientation of their precession axes. Commutation of the sequential order of these two flippers leads to an observable modification of the polarization of the outgoing beam, which is exactly the effect of the noncommuting property of components of the Pauli spin operator ($\sigma_x$ and $\sigma_z$ in our specific arrangement). We interpret our experimental results as an interference effect between two mutually orthogonal spin states and by the accumulation of different phase shifts upon reversal of the sequence of spin rotations. It is shown that the phase shift caused by the two successive spin rotations carries specific information about the intermediate state and finally results in the well-known commutation relation of Pauli spin operators. We discuss this intrinsic phase shift with respect to its geometrical and dynamical components. This intrinsic phase is significant only in quantum mechanics and allows for a clear distinction between quantum-mechanical and classical description of the motion of particles.

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## II. NEUTRON POLARIMETRIC SPINOR-ROTATION EXPERIMENTS

### A. Theoretical background

Manipulations of the wave function of spin-$\frac{1}{2}$ particles are conveniently carried out using the two-component spinor formalism. Neutron spin rotation around an axis $\hat{\boldsymbol{\alpha}}$ by an angle $\alpha$ is described by the unitary operator [1]

$$U_R = \exp(-i\boldsymbol{\sigma}\cdot\boldsymbol{\alpha}/2), \tag{1}$$

where $\boldsymbol{\alpha} = \alpha\hat{\boldsymbol{\alpha}}$ is the rotation vector. We may think of variously oriented rotations, and hence let us consider two successive rotations about different axes. In general rotations about different axes fail to commute. This is true for spinor rotations as well as for classical rotations. The latter, however, are described by O(3), whereas rotations in quantum mechanics are described by angular momentum operators. For instance, spinor rotations of spin-$\frac{1}{2}$ particles obey SU(2) [31]. In the simplest case the noncommutation of spinor rotations of spin-$\frac{1}{2}$ particles can be attributed to the noncommuting features of Pauli matrices. For instance, the commutation and the anticommutation relations between $\sigma_x$ and $\sigma_z$ are given by

$$[\sigma_z, \sigma_x] = \sigma_z\sigma_x - \sigma_x\sigma_z = 2i\sigma_y \tag{2}$$

and

$$\sigma_x\sigma_z + \sigma_z\sigma_x = 0. \tag{3}$$

Although this commutation relation is just a mathematical fact, we will see its actual manifestation when we consider physical rotations of quantum objects.

Experimentally neutron spin rotation can be accomplished by virtue of a magnetic field. Then $\alpha$ corresponds to the Larmor precession angle accumulated by the neutron magnetic moment upon passage through that field. If this precession angle is set to $\pi$, the operator $U_R$ defined by Eq. (1) becomes $-i\boldsymbol{\sigma}\cdot\hat{\boldsymbol{\alpha}}$. Here we assume that the neutrons pass through two successive magnetic fields which are oriented in different directions. Let us take two orientations of the magnetic fields, $\hat{\boldsymbol{\alpha}}_A = (1,0,0)$ for a field in the $+\hat{\mathbf{x}}$ direction and $\hat{\boldsymbol{\alpha}}_B = (\cos\beta, 0, \sin\beta)$ for a field in the $\hat{\mathbf{x}}$-$\hat{\mathbf{z}}$ plane. Then the corresponding $\pi$-rotation operators, denoted by $A$ and $B$, are given by

$$A \equiv U_R(\hat{\boldsymbol{\alpha}}_A) = -i\sigma_x \tag{4a}$$

and

$$B \equiv U_R(\hat{\boldsymbol{\alpha}}_B) = -i(\sigma_x\cos\beta + \sigma_z\sin\beta). \tag{4b}$$

The combined spin rotation operator can be denoted either by $AB$ or by $BA$, depending on the actually chosen sequential order of these two different $\pi$ rotations. The combined operator $AB$ has the form

$$\begin{aligned} AB &= -(\sigma_x^2\cos\beta + \sigma_x\sigma_z\sin\beta) \\ &= -(\mathbf{1}\cos\beta - i\sigma_y\sin\beta) \\ &\text{or } -\begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}, \end{aligned} \tag{5a}$$

whereas its counterpart with a commuted sequence of $A$ and $B$ is calculated as

$$\begin{aligned} BA &= -(\sigma_x^2\cos\beta + \sigma_z\sigma_x\sin\beta) \\ &= -(\mathbf{1}\cos\beta + i\sigma_y\sin\beta) \\ &\text{or } -\begin{bmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{bmatrix}. \end{aligned} \tag{5b}$$

Because of the noncommutation of the Pauli matrices $\sigma_x$ and $\sigma_z$, the commutation of $A$ and $B$ obviously should lead to a modification of the final neutron spin state, i.e., to a different polarization of the transmitted neutron beam.

Usually the spin state $|s\rangle$ of a neutron is described by a normalized spinor

$$|s\rangle = \begin{bmatrix} \cos\left(\dfrac{\theta}{2}\right) \\ e^{i\phi}\sin\left(\dfrac{\theta}{2}\right) \end{bmatrix}, \tag{6}$$

where $\theta$ and $\phi$ are the polar and azimuthal angles with respect to a chosen quantization axis ($z$ axis). The polarization vector $\mathbf{P} \equiv \langle s|\boldsymbol{\sigma}|s\rangle$ of the neutron beam is defined as the expectation value of the Pauli spin operator. Let $|s\rangle$ denote the initial spin state. Then, after passage through two successive magnetic field regions oriented along $\hat{\boldsymbol{\alpha}}_A$ and $\hat{\boldsymbol{\alpha}}_B$, respectively, the neutron spinor changes to

$$|s_{BA}\rangle \equiv BA|s\rangle = -\begin{bmatrix} \cos\beta\cos\left(\dfrac{\theta}{2}\right) + e^{i\phi}\sin\beta\sin\left(\dfrac{\theta}{2}\right) \\ -\sin\beta\cos\left(\dfrac{\theta}{2}\right) + e^{i\phi}\cos\beta\sin\left(\dfrac{\theta}{2}\right) \end{bmatrix}. \tag{7}$$

On the other hand, when the orientations of the magnetic fields are mutually commuted and hence the neutron beam propagates through the magnetic field regions oriented at first to $\hat{\boldsymbol{\alpha}}_B$ and then to $\hat{\boldsymbol{\alpha}}_A$, the spin state is modified into

$$|s_{AB}\rangle \equiv AB|s\rangle = -\begin{bmatrix} \cos\beta\cos\left(\dfrac{\theta}{2}\right) - e^{i\phi}\sin\beta\sin\left(\dfrac{\theta}{2}\right) \\ \sin\beta\cos\left(\dfrac{\theta}{2}\right) + e^{i\phi}\cos\beta\sin\left(\dfrac{\theta}{2}\right) \end{bmatrix}. \tag{8}$$

The corresponding final polarization vectors of these two beams are given by

$$\mathbf{P}_{BA} = \langle s_{BA}|\boldsymbol{\sigma}|s_{BA}\rangle = \begin{pmatrix} \sin\theta\cos\phi\cos2\beta - \cos\theta\sin2\beta \\ \sin\theta\sin\phi \\ \sin\theta\cos\phi\sin2\beta + \cos\phi\cos2\beta \end{pmatrix} \tag{9a}$$

and

$$\mathbf{P}_{AB} = \langle s_{AB}|\boldsymbol{\sigma}|s_{AB}\rangle = \begin{pmatrix} \sin\theta\cos\phi\cos2\beta + \cos\theta\sin2\beta \\ \sin\theta\sin\phi \\ -\sin\theta\cos\phi\sin2\beta + \cos\theta\cos2\beta \end{pmatrix}. \tag{9b}$$

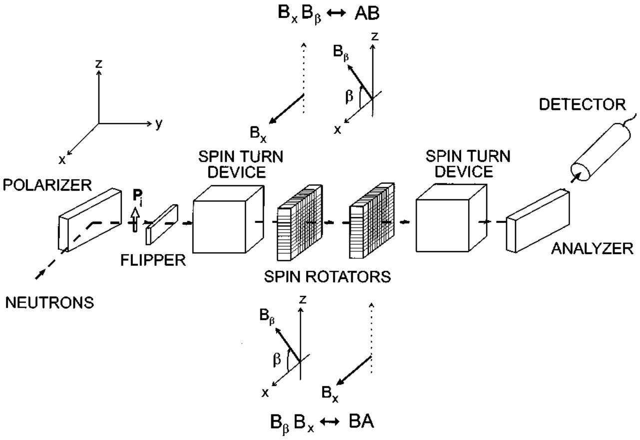

FIG. 1. Schematic sketch of the experimental setup. Two spin-turn devices allow for an arbitrary orientation of the incident and the finally analyzed polarization vector. The spin turn rotators are two successive $\pi$ spin flippers with stationary magnetic fields oriented in directions $\hat{\boldsymbol{\alpha}}_A$ and $\hat{\boldsymbol{\alpha}}_B$, respectively. Their sequential order is mutually exchanged to realize the commuted operators *AB* and *BA*.

For example, if the incident neutrons are polarized in the $+\hat{\mathbf{z}}$ direction, i.e., $\mathbf{P}_i=(0,0,1)$, the final polarization vectors given by Eqs. (9) for $\phi=\pi$ reduce to $\mathbf{P}_{BA}=(-\sin 2\beta,0,\cos 2\beta)$ and $\mathbf{P}_{AB}=(\sin 2\beta,0,\cos 2\beta)$, respectively. Here it is worth noting that, while the *y* and *z* components of $\mathbf{P}_{BA}$ and $\mathbf{P}_{AB}$ are equal, the noncommutation of $\sigma_x$ and $\sigma_z$ causes their *x* components to differ from each other. In the above example, only the matrix elements of the first column of the combined operators *AB* and *BA*, defined by Eq. (5), are responsible for the results. In order to check also the matrix elements of the second column, one requires a differently oriented polarization vector of the incident beam, e.g., in the $-\hat{\mathbf{z}}$ direction.

Of course, classical rotations of the polarization vector would be sufficient to end up at the same final polarizations. However, the presented quantum-mechanical description of spinor rotations of spin-$\frac{1}{2}$ particles goes far beyond a classical treatment. If one assumes, for instance, $\mathbf{P}_i=(0,0,1)$ and $\beta=\pi/2$, the spinor wave functions are not the same after passage through two successive magnetic fields oriented in different directions, whereas the resultant polarization vectors remain identical. Therefore, the presented textbooklike results of different final polarization states due to mutual interchange of the two magnetic field regions traversed by the neutron beam can be attributed, indeed, to the quantum-mechanical feature of noncommuting Pauli matrices.

### B. Experimental results

The experiments were carried out at a neutron polarimetry facility installed at the 250-kW TRIGA reactor in Vienna, usually serving for three-dimensional neutron depolarization studies of magnetic domain structures. A schematic view of the experimental setup is shown in Fig. 1. The incident neutron beam is monochromatized ($\lambda=1.53$ Å) and polarized (average degree of polarization $|\mathbf{P}_i|\cong 0.95$) by a magnetically saturated Heusler-alloy single crystal. Its diameter is confined to 4 mm by a Cd diaphragm. The initial polarization vector $\mathbf{P}_i$ is perpendicular to the beam trajectory and defines the $+\hat{\mathbf{z}}$ direction. A second Heusler crystal is used to analyze the final polarization $\mathbf{P}_f$. Depolarization of the neutron beam is minimized by permanent magnetic guide fields distributed along its trajectory wherever necessary. Specially designed magnet coil configurations, whose construction details are not of importance here and which are attached immediately in front and behind a soft-magnetic iron shielding box, together with an appropriate spin flipper serve to orient the incident and to analyze the final polarization vectors in any of the three directions of space. (For the sake of clarity both the magnetic shielding and the guide fields are omitted in the drawing.) These spin-turn devices thus enable one to measure successively all nine elements of the $(3\times 3)$ polarization transfer matrix $\underline{\underline{M}}$ which describes the change of the polarization vector during the neutron's propagation through the sample under investigation according to the relation $\mathbf{P}_f=\underline{\underline{M}}\mathbf{P}_i$.

However, in our experiment instead of a sample two separately tunable spin rotators are mounted within the magnetically shielded soft iron box in order to realize the rotation operators $U_R(\hat{\boldsymbol{\alpha}}_A)$ and $U_R(\hat{\boldsymbol{\alpha}}_B)$ independently from each other. Each of these spin rotators consists of two mutually orthogonal coils wound on a common square-shaped frame to produce magnetic fields both in the $\pm\hat{\mathbf{x}}$ and the $\pm\hat{\mathbf{z}}$ direction. By varying the ratio of the coil currents, the directions of the total fields can be rotated in the $\hat{\mathbf{x}}$-$\hat{\mathbf{z}}$ plane, thereby defining the directions $\hat{\boldsymbol{\alpha}}_A$ and $\hat{\boldsymbol{\alpha}}_B$, under the constraint that constant spin rotations by an angle $\pi$ have to be maintained.

In the first experiment the polarization of the incident neutron beam was set in the $+\hat{\mathbf{z}}$ direction. In order to show the noncommutation of the operators *A* and *B*, defined by Eqs. (5), the spin-rotator devices were adjusted so that the upstream one represents $U_R(\hat{\boldsymbol{\alpha}}_A)$ and the downstream one $U_R(\hat{\boldsymbol{\alpha}}_B)$, and vice versa. Neutron intensities were recorded as a function of the angle $\beta$ for polarization analysis in all three directions of space, which corresponds to measurements of the two elements $M_{ZX}$ and $M_{ZZ}$ of the polarization transfer matrix $\underline{\underline{M}}$. Figure 2 shows the experimental data together with theoretically predicted curves with least-squares fits of mean intensity and oscillation amplitude. Here one can see that, as theory predicts, commutation of the two spin rotation operators has no influence on the $\beta$ dependence of the *z*-polarization components but leads to inverse modulation of the *x* component. Although the emerging polarization should have no *y* component, a slight residual intensity modulation persisted upon variation of $\beta$ when the matrix element $P_{zy}$ was measured. This could be identified as being due to a misalignment of the polarization analysis direction from the $+\hat{\mathbf{y}}$ axis by an angle of 1.3°.

As mentioned already, with the polarization of the incident beam oriented in the $+\hat{\mathbf{z}}$ direction, the experimental results are completely attributed to the first column matrix elements of the commuted operators *AB* and *BA*. In order to show the consequences from the matrix elements of the second column, it is necessary to orient the incident polarization vector in the $-\hat{\mathbf{z}}$ direction. In Fig. 3 the corresponding intensity modulations are plotted in the same manner as in Fig. 2. Here, too, a quantitative agreement between the experimental results and the theoretical predictions is evident.

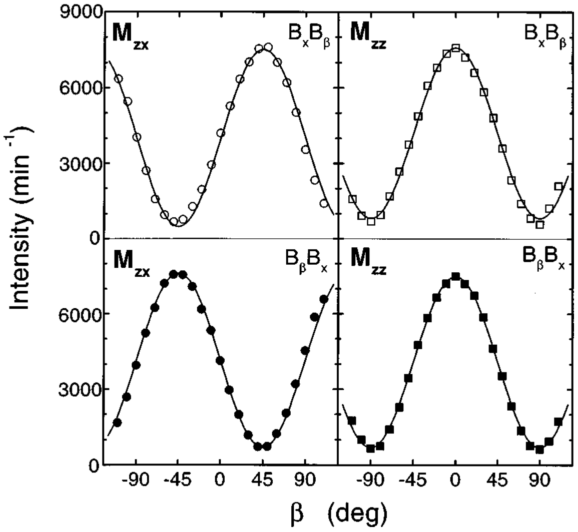

FIG. 2. Neutron intensity as a function of the inclination angle of the magnetic field direction for two commuted operations. The full lines correspond to the theoretical predictions. The incident polarizations were oriented in the $+\hat{\mathbf{z}}$ direction and the final polarization was successively analyzed along the $+\hat{\mathbf{x}}$ and $+\hat{\mathbf{z}}$ directions, thus yielding the elements $M_{ZX}$ and $M_{ZZ}$ of the polarization transfer matrix $\underline{\underline{M}}$ (see text).

### C. Different phase shifts due to commuted spinor rotations

The experiments have shown that, when the incident neutrons polarized either in the $+\hat{\mathbf{z}}$ or the $-\hat{\mathbf{z}}$ direction pass through two successive $\pi$ spin rotators with differently oriented precession axes, the final polarization vectors were given by $\mathbf{P}_{AB}=(\pm\sin 2\beta,0,\pm\cos 2\beta)$ and $\mathbf{P}_{BA}=(\mp\sin 2\beta,0,\pm\cos 2\beta)$. According to the Larmor interference concept of neutron spin rotation [18], we interpret these results as a consequence of different phase shifts induced by the actions of the operators $AB$ and $BA$. Let us assume two orthogonal spin states, $|+y\rangle$ and $|-y\rangle$. These are eigenstates of $AB$ and $BA$, and the operations of $AB$ and $BA$ on these states yield

$$\begin{aligned} AB|\pm y\rangle &= -e^{\mp i\beta}|\pm y\rangle, \\ BA|\pm y\rangle &= -e^{\pm i\beta}|\pm y\rangle. \end{aligned} \tag{10}$$

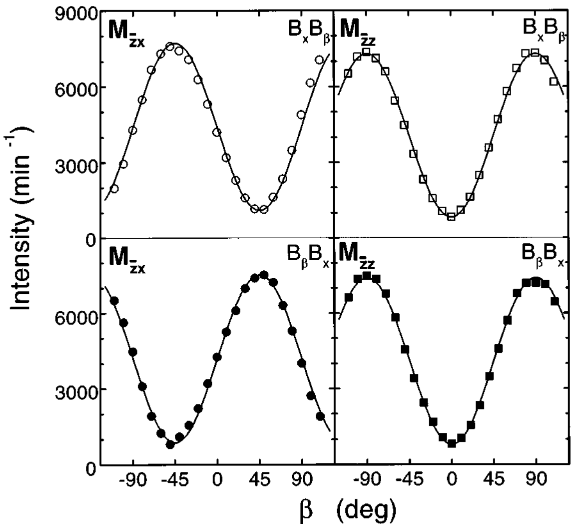

FIG. 3. As in Fig. 2, but with the incident polarization in the $-\hat{\mathbf{z}}$ direction thus yielding the elements $M_{\bar{Z}X}$ and $M_{\bar{Z}Z}$ of the polarization transfer matrix $\underline{\underline{M}}$. Here the effects from the second column of the operators $AB$ and $BA$ are observed [see Eqs. (5) and (9)].

Intuitively one might assume that the different phase shifts $\pm\beta$ induced by the operators $AB$ and $BA$ should not be accessible via neutron polarimetry. However, as has been demonstrated explicitly by polarized neutron perfect crystal interferometry [14,15], superposition of a spin-up and a spin-down state results in a new state which is oriented perpendicular to its constituents. In that case the final polarization depends upon the phase difference between two orthogonal spin states. Thus, by considering a state $|\pm z\rangle$ as a superposition of $|+y\rangle$ and $|-y\rangle$ states, it becomes clear why a neutron polarimetric experiment, which allows full control over all three spatial components of both the incident and the final polarization vector, yields information about phase differences between these states. According to Eq. (10) one obtains the final states for an incident spin state $|+z\rangle$ as

$$\begin{aligned} AB|+z\rangle &= \frac{1}{\sqrt{2}}(AB|+y\rangle + AB|-y\rangle) \\ &= -\frac{1}{\sqrt{2}}e^{-i\beta}(|+y\rangle + e^{2i\beta}|-y\rangle) \end{aligned} \tag{11a}$$

and, correspondingly,

$$BA|+z\rangle = -\frac{1}{\sqrt{2}}e^{i\beta}(|+y\rangle + e^{-2i\beta}|-y\rangle). \tag{11b}$$

These equations show that the spin vectors of the states affected by $AB$ and $BA$ rotate in the $\hat{\mathbf{x}}$-$\hat{\mathbf{z}}$ plane by an angle $\pm 2\beta$ (measured from the $+\hat{\mathbf{x}}$ direction). Thus, while the $y$ and the $z$ components of both final polarization vectors $\mathbf{P}_{AB}$ and $\mathbf{P}_{BA}$ are identical, their $x$ components differ due to the different phase shifts $\pm\beta$ of the states $|\pm y\rangle$ that are induced by the operators $AB$ and $BA$. Notice that both of these operators turn the initial states $|\pm y\rangle$ back to the same states but that the phase of the final states depends on the intermediate states, that is, on the chosen trajectory in spin space. Evidently there is a close connection between this spin transfer and the concept of geometric phases [20].

## III. NONCOMMUTATION OF PAULI SPIN OPERATORS DUE TO INTRINSIC PHASE SHIFTS

### A. Spin-$\frac{1}{2}$ particles

The simplest way to deal with spin commutation relations is to consider massive particles with spin quantum number $\frac{1}{2}$, such as, e.g., electrons or neutrons. The complete spin wave function $|\Psi\rangle$ of spin-$\frac{1}{2}$ particles contains not only the "pure" spinor $|s\rangle$, defined already by Eq. (6), but also the intrinsic phase $\xi$,

$$|\Psi\rangle = e^{i\xi}|s\rangle = |\xi,\theta,\phi\rangle. \tag{12}$$

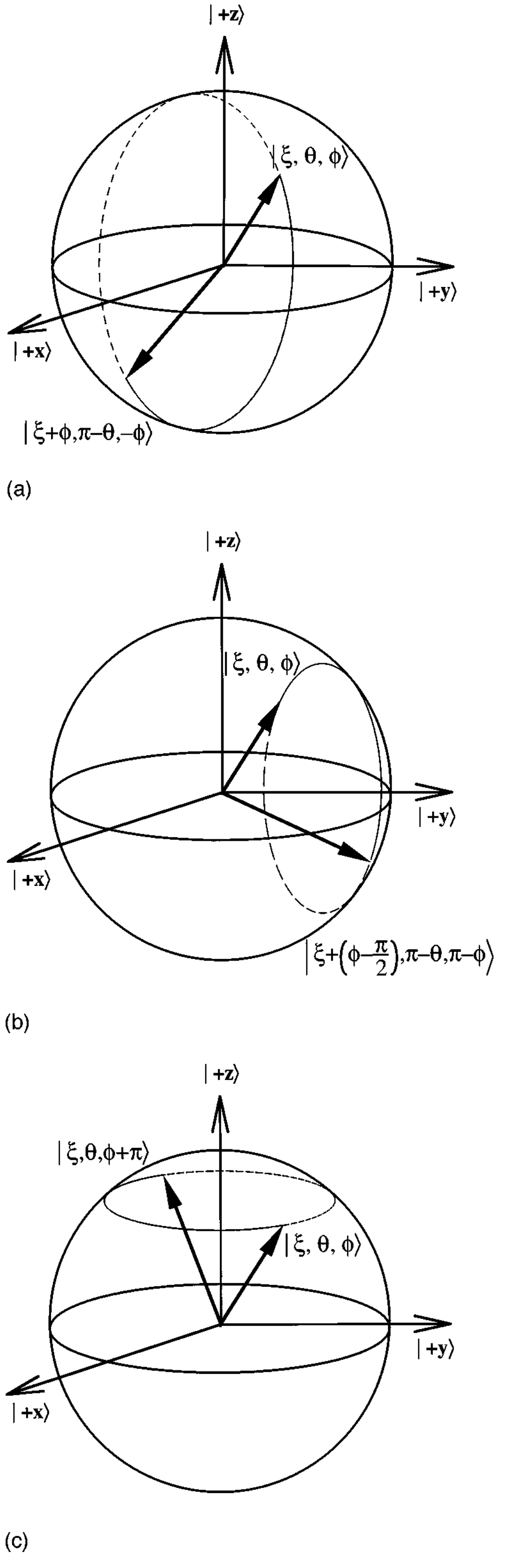

FIG. 4. $\sigma_i$ operations on the wave function $|\Psi\rangle$ $(=|\xi,\theta,\phi\rangle)$. (a) $\sigma_x$ operates as a rotation along the $x$ axis and the additional phase shift $\phi$. (b) $\sigma_y$ operates as a rotation along the $y$ axis and the additional phase shift $\phi-\pi/2$. (c) $\sigma_z$ operates as a rotation along the $z$ axis without any additional intrinsic phase shift.

It should be mentioned here that the choice of coordinate axes is arbitrary and that the intrinsic phase $\xi$ and the spinor $|s\rangle$ depend on the chosen coordinate which, in turn, gives the spin wave function $|\Psi\rangle$ as a whole. However, the associated polarization vector

$$\mathbf{P}=\langle\Psi|\boldsymbol{\sigma}|\Psi\rangle=(\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta) \qquad (13)$$

is independent of the intrinsic phase $\xi$. For that reason in the usual description of spin-$\frac{1}{2}$ particles this overall phase factor is omitted. Instead of the generalized spinor $|\Psi\rangle$, the spin state then is represented by the reduced spinor $|s\rangle$. In the so-called Poincaré sphere representation [32] two quantities, the polar angle $\theta$ and the azimuthal angle $\phi$, are sufficient to describe this ''pure'' spin state $|s\rangle$.

Consequently, it must be emphasized that even when two spin states are the same, their wave functions are not necessarily identical. However, since in conventional intensity measurements the detection probability is proportional to the squared modulus of the wave function, the intrinsic phase plays no practical role in describing the motion of particles. The change of the intrinsic phase is detectable only via an interference experiment by coherent superposition with a reference wave function.

### B. Action of Pauli spin operators

To take the intrinsic phase into account, we use the full spin wave function $|\Psi\rangle(=|\xi,\theta,\phi\rangle)$ in representing the spin state transformations induced by the action of the Pauli spin operators. When $\sigma_x$ acts upon this wave function, the transformed state is calculated to be

$$\begin{aligned}\sigma_x|\xi,\theta,\phi\rangle&=\sigma_x e^{i\xi}\left[\cos\left(\frac{\theta}{2}\right)|+z\rangle+e^{i\phi}\sin\left(\frac{\theta}{2}\right)|-z\rangle\right]\\&=e^{i(\xi+\phi)}\left[\cos\left(\frac{\pi-\theta}{2}\right)|+z\rangle\right.\\&\quad\left.+e^{-i\phi}\sin\left(\frac{\pi-\theta}{2}\right)|-z\rangle\right]\\&=|\xi+\phi,\pi-\theta,-\phi\rangle. \qquad (14)\end{aligned}$$

The Poincaré sphere representation of this action is shown in Fig. 4(a). Besides a rotation around the $x$ axis, $\sigma_x$ causes an overall phase shift which depends on the azimuthal angle $\phi$ of the initial spin state. We will show that it is just this overall phase shift which is responsible for the noncommutation properties of the Pauli spin operator.

Likewise the transformation of the total spin wave function $|\Psi\rangle$ under the action of $\sigma_y$ is obtained as

$$\begin{aligned}\sigma_y|\xi,\theta,\phi\rangle&=\sigma_y e^{i\xi}\left[\cos\left(\frac{\theta}{2}\right)|+z\rangle+e^{i\phi}\sin\left(\frac{\theta}{2}\right)|-z\rangle\right]\\&=e^{i[\xi+\phi-(\pi/2)]}\left[\cos\left(\frac{\pi-\theta}{2}\right)|+z\rangle\right.\\&\quad\left.+e^{i(\pi-\phi)}\sin\left(\frac{\pi-\theta}{2}\right)|-z\rangle\right]\\&=|\xi+\phi-(\pi/2),\pi-\theta,\pi-\phi\rangle. \qquad (15)\end{aligned}$$

This transformation is illustrated in Fig. 4(b). There the rotation takes place around the $y$ axis and the additional phase shift, $(\phi-\pi/2)$ in this case, again depends on the azimuthal angle $\phi$. Notice, however, that this phase shift differs by $-\pi/2$ from that caused by $\sigma_x$.

Finally, it is immediately seen from

$$\sigma_z|\xi,\theta,\phi\rangle=|\xi,\theta,\phi+\pi\rangle \tag{16}$$

that $\sigma_z$ acts only as rotation around the $z$ axis and that no phase shift occurs for the chosen set of $(\pm z)$-basis states. The Poincaré sphere representation of this transformation is shown in Fig. 4(c).

### C. Commutation relation via intrinsic phase shift

In analogy to the preceding subsection, the transformation of the wave function $|\Psi\rangle$ $(=|\xi,\theta,\phi\rangle)$ under the successive operation of $\sigma_x$ and then $\sigma_y$ can be calculated as

$$\begin{aligned}\sigma_y\sigma_x|\xi,\theta,\phi\rangle\\ &=\sigma_y|\xi+\phi,\pi-\theta,-\phi\rangle\\ &=\left|(\xi+\phi)+\left(-\phi-\frac{\pi}{2}\right),\pi-(\pi-\theta),\pi-(-\phi)\right\rangle\\ &=\left|\xi-\frac{\pi}{2},\theta,\pi+\phi\right\rangle\\ &=e^{-i(\pi/2)}\sigma_z|\xi,\theta,\phi\rangle.\end{aligned} \tag{17}$$

Using again the Poincaré sphere representation, this successive operation is shown in Fig. 5(a). According to Eq. (16), $\sigma_z$ yields no overall phase-shift contribution. Hence it is clearly seen that the successive operation of $\sigma_x$ and $\sigma_y$ causes a fixed intrinsic phase shift of the wave function of exactly $-\pi/2$.

On the other hand, commutation of the sequence of $\sigma_x$ and $\sigma_y$ yields the transformation [see also Fig. 5(b)]

$$\begin{aligned}\sigma_x\sigma_y|\xi,\theta,\phi\rangle\\ &=\sigma_x\left|\xi+\left(\phi-\frac{\pi}{2}\right),\pi-\theta,\pi-\phi\right\rangle\\ &=\left|\left(\xi+\phi-\frac{\pi}{2}\right)+(\pi-\phi),\pi-(\pi-\theta),-(\pi-\phi)\right\rangle\\ &=\left|\xi+\frac{\pi}{2},\theta,-\pi+\phi\right\rangle=e^{i(\pi/2)}\sigma_z|\xi,\theta,\phi\rangle.\end{aligned} \tag{18}$$

Here the $2\pi$ periodicity of the state with respect to $\phi$ has been taken into account. The total phase shift induced by this commuted sequence of spinor transformations amounts to exactly $+\pi/2$. It should be emphasized that this difference from the previously obtained value $-\pi/2$ arises essentially from the different azimuthal angle of the intermediate state of these two successive transformations.

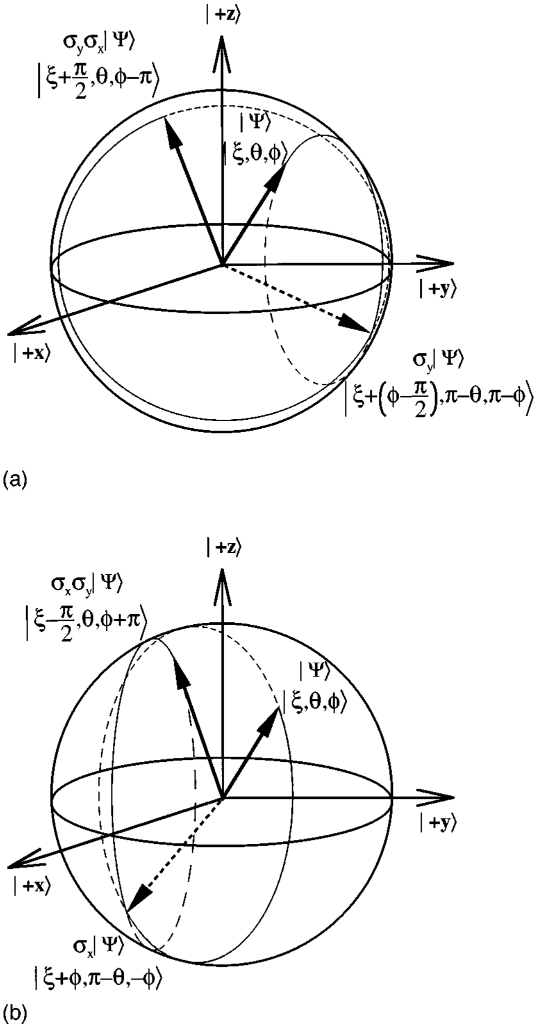

FIG. 5. $\sigma_i\sigma_j$ operation on the wave function $|\Psi\rangle$ $(=|\xi,\theta,\phi\rangle)$. (a) $\sigma_x\sigma_y$ is equivalent to the operation $\sigma_z$ in addition to an intrinsic phase shift $+\pi/2$. (b) $\sigma_y\sigma_x$ is equivalent to the operation $\sigma_z$ plus an intrinsic phase shift $-\pi/2$.

A comparison of Eqs. (17) and (18) immediately leads to the relations

$$\sigma_y\sigma_x=e^{-i(\pi/2)}\sigma_z \tag{19a}$$

and

$$\sigma_x\sigma_y=e^{i(\pi/2)}\sigma_z. \tag{19b}$$

Finally, we can get the explicit form of the commutator $[\sigma_x,\sigma_y]$ as follows:

$$\begin{aligned}[\sigma_x,\sigma_y]&=e^{i(\pi/2)}\sigma_z-e^{-i(\pi/2)}\sigma_z=(e^{i(\pi/2)}-e^{-i(\pi/2)})\sigma_z\\ &=2i\sigma_z.\end{aligned} \tag{20}$$

From these formulas it becomes obvious that the commutation relations of the Pauli spin operator are a direct consequence of these intrinsic phase shifts $\pm\pi/2$. While the spin states are exactly the same after the operations of $\sigma_x\sigma_y$ and $\sigma_y\sigma_x$, respectively, the associated intrinsic phase shifts are different and result in the commutation relation. Of course, all the other commutation relations with $\sigma_i\sigma_j$ $(i\neq j)$ can be derived as well in the same manner.

## IV. DISCUSSION

A commutation of the operators $A$ and $B$ has no influence on the measured intensity pattern when the $z$-polarization component of the emerging beam is analyzed, and in all cases the $y$ component remains zero. This can be seen not only from the equations derived in the theory. It can be understood also intuitively without any knowledge of spinor rotations of spin-$\frac{1}{2}$ particles, although the spinor description is a more general concept than the classical description of rotation. One should consider that the incident $\pm\hat{\mathbf{z}}$ polarized beam is only affected by two successive $\pi$ rotations around two different axes, both of which are always in the $\hat{\mathbf{x}}$-$\hat{\mathbf{z}}$ plane. Consequently, the final polarization vector also is permanently confined to the $\hat{\mathbf{x}}$-$\hat{\mathbf{z}}$ plane.

The state of massive particles of spin is represented by the wave function which is derived by combining the intrinsic phase $\xi$ and the spin polarization state $|s\rangle$. While the polarization vector of each particle is completely defined with two parameters, i.e., polar angle $\theta$ and azimuthal angle $\phi$, the wave function is identified with three parameters: $\theta$, $\phi$, and the intrinsic phase $\xi$. However, since the detection probability is given by the absolute square of the wave function, the information on the intrinsic phase of the quantum state gets lost, spoiling the difference between a classical description of the motion of particles.

The complete description on the quantum state of massive spin-$\frac{1}{2}$ particles requires the wave function $|\Psi\rangle$ $(=|\xi,\theta,\phi\rangle)$. Equations (14)–(16) have shown that the components $\sigma_i$ of the Pauli spin operators cause characteristic shifts of the intrinsic phase $\xi$ of the wave function in addition to their respective rotation of the pure spin state. Two of these phase shifts depend on the azimuthal angle of the operated spin state. Furthermore, it was shown that the commutation relation of Pauli spin operators, say $\sigma_x$ and $\sigma_y$, results from the $\pm\pi/2$ shift of the intrinsic phase after two successive operations. From the occurrence of a phase jump $\pi$ upon commuting the sequence of the two spin operators, it becomes evident that the intrinsic phase carries information about the intermediate state between the successive transformations. In a mathematical sense this information transfer is just another interpretation of the non-Abelian character of Pauli spin matrices $\sigma_i$ [33].

If, for instance, the initial spin state is $|+z\rangle$, anticommutation of $\sigma_x$ and $\sigma_y$ results from the fact that the geometric phase factors associated with the operations $\sigma_x\sigma_y$ and $\sigma_y\sigma_x$ differ by $\pi$ whereas no dynamical phase contributions arise [Fig. 6(a)]. On the other hand, if the initial state is $|+x\rangle$, both operations trace identical paths on the Poincaré sphere [Fig. 6(b)]. Thus, there is no geometric phase difference, but now the dynamical phases accumulated along these two closed trajectories differ exactly by a value $\pi$ according to the operation of $\sigma_x$ on the $|+x\rangle$ and the $|-x\rangle$ spin state, before and after the $\pi$ flip by $\sigma_y$ has taken place. In the general case, when the initial spin state coincides with neither the $|\pm z\rangle$ nor the $|\pm x\rangle$ state, $\sigma_x\sigma_y$ and $\sigma_y\sigma_x$ will trace different paths on the Poincaré sphere. Figure 6(c) shows two paths from the initial state $|s\rangle$ through the intermediate state $|s'\rangle$ to the final one $|s''\rangle$; shaded parts refer to the geometrical phase factor. Thus, dynamical as well as geometric phase contribution will be accumulated along both trajectories. However, their sum will always be balanced to yield a difference of exactly $\pi$, thereby leading to anticommutation of $\sigma_x$ and $\sigma_y$. It is worth emphasizing here that anticommutation of any two Pauli spin operators, $\sigma_i$ and $\sigma_j$ $(i\neq j)$, can be attributed to the balanced difference of the geometrical and dynamical phase shifts.

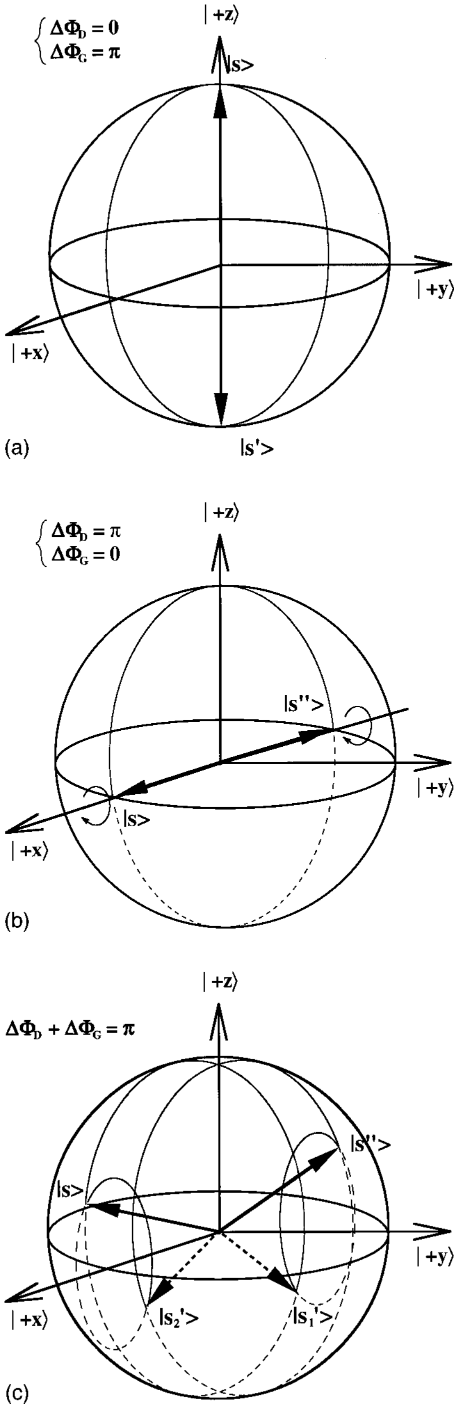

FIG. 6. Anticommutation relation of $\sigma_x$ and $\sigma_y$ due to the geometrical and dynamical phase factors. (Shaded parts refer to the geometric phase shift contributions.) (a) If the initial spin polarization state $|s\rangle$ is $|+z\rangle$, the anticommutation relation results from the geometrical phase factor difference of exactly $\pi$. (b) If the initial spin state is $|+x\rangle$, it results from a dynamical phase difference of $\pi$. (c) For arbitrary initial state it follows from the balanced sum $\pi$ of the differences of the geometric and dynamical phase.

The commutator $[\sigma_x,\sigma_y]\equiv\sigma_x\sigma_y-\sigma_y\sigma_x$ is a good example of a cyclic evolution where the system can be regarded as being split and subsequently recombined again to the same state after an evolution under two separate Hamiltonians has taken place. Thus, it is worth mentioning that the difference $\pi$ of the intrinsic phase of the wave function after transformations $\sigma_x\sigma_y$ and $\sigma_y\sigma_x$ is related to the geometrical and/or dynamical phase-shift components of the spinor evolution. It has been shown already that a specific geometrical phase factor caused by two identical but differently oriented successive $\pi$ flippers will lead to their anticommutability [$\sigma_x\sigma_y+\sigma_y\sigma_x=0$, see Eq. (3)] [27] and that neutron interferometry is able to identify the geometric and dynamical phase contributions of such spinor transformation [27–29,34]. However, as shown in Fig. 6, the present analysis on the noncommutation of two successive Pauli spin operators is more general than the preceding example described in Ref. [34].

In the interferometric experiment [29], the initial spinor $|+z\rangle$ was brought to the final one $|-z\rangle$ via $-\sigma_p\sigma_q$ and $-\sigma_q\sigma_p$, and the anticommutativity of orthogonal components of the Pauli spin operator was observed due to a pure geometric phase shift by 180°. In contrast, the polarization change in the present polarimetric experiments results from the *difference* $\pm 2\beta$ of the geometric phases via operation *AB* or *BA* for spinors $|\pm y\rangle$, ranging from $-220°$ to $220°$. That is, the commutated operators, both *AB* and *BA*, bring the initial spinors $|\pm y\rangle$ to the same final spinors and each of them induce a geometric phase $+\beta$ or $-\beta$ (due to different ''orange slices'') for corresponding spinors [see Eq. (9)]; these different geometric phases lead to the polarization change [see Eq. (10)]. It should be emphasized here that the obtained results in the present polarimetric experiments reflect more general consequences of the noncommutation of the Pauli spin operator than its anticommutativity.

The Poincaré sphere description is available to represent the spin polarization state [32]. While the spin state of each particle, which belongs to the pure state, lies on the sphere, in general the mixed state of a non-completely-polarized ensemble of particles lies within the sphere. Although such a description seems to be quite general, it omits the intrinsic phase. Since the final spin states after the transformations $\sigma_x\sigma_y$ and $\sigma_y\sigma_x$ are exactly the same, this omission allows no adequate interpretation of spin commutation relations. Thus the justification of a Poincaré sphere representation of spin states is restricted to cases where the intrinsic phase need not to be taken into account.

## V. CONCLUSIONS

In summary, we have exploited neutron spin polarimetry to demonstrate the noncommuting behavior of spinor rotations. In changing the sequential order of two $\pi$ spin-turn devices with differently oriented precession axes, the transmitted neutron beam ended up with different *x* components of their respective polarization vectors if the initial beam was polarized parallel or antiparallel to the *z* direction. The polarization difference observed in our experiment can be attributed to the anticommutation properties of the two Pauli spin matrices $\sigma_x$ and $\sigma_y$. In addition, the noncommutation of these spin operators can be interpreted also as a consequence of the different shifts of the intrinsic phase of the wave function when they are commuted. These shifts of the intrinsic phase by an angle $\pm\pi/2$ carry the information about the intermediate spin state between two successive transformations. The induced phase shifts contain both geometric and dynamical contributions. These are always balanced to yield a total difference of exactly $\pi$, which results in the familiar anticommuting relations of Pauli spin matrices. Obviously the intrinsic phase of the wave function, which is usually considered as being of no significance in particular if the motion of particles is to be described, is essential for a complete quantum-mechanical treatment of spinor transformations.

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